A FUNCTORIAL CHARACTERIZATION OF Tor FOR NOETHERIAN RINGS OF GLOBAL DIMENSION 1

BY
DAVID NEWELL

0. **Introduction.** Let R be a ring and let ${}_R\boldsymbol{M}$ denote the category of left R-modules and left R-homomorphisms. A functor $G\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$, where $\boldsymbol{Ab}$ is the category of abelian groups, is called a Tor-*functor* if there is a right R-module N for which the functor $\mathrm{Tor}_1^R(N, \quad)\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ is naturally equivalent to G.

In this paper, we show that if R is a noetherian ring of global dimension 1 (for example, if R is a Dedekind domain, or a ring of triangular matrices over a field), then the class of Tor-functors is characterized as that class of functors G for which G is half exact, G preserves direct limits, and $G(R)=0$.

In [2], Auslander shows that if R is noetherian and $F\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ is a coherent half exact functor which preserves direct limits, then there is a right R-module N and an exact sequence of functors

$$\mathrm{Tor}_2^R(N, \quad) \to F(R) \otimes_R \to F \to \mathrm{Tor}_1^R(N, \quad) \to 0.$$

In §1, we recall the definition of coherent functors and some of their properties from [2]. We also show that every half exact functor is a filtered limit of half exact coherent functors ("filtered limit" being our term for a generalization of direct limits defined in [1]).

In §2, we show that if R is noetherian and $G\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ is a half exact functor which preserves direct limits, then there is an exact sequence for G similar to the one above, provided that a filtered limit of Tor functors is a Tor functor. (If R is of global dimension 1, this condition is satisfied, and our characterization of Tor functor follows immediately.)

In §3, we extend Auslander's exact sequence to the class of coherent rings, namely rings for which the dual of projective modules are flat.

In §4, we let R be a coherent ring and $G\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ be a half exact functor which preserves direct limits, and obtain a result similar to that of §2. We also compare the coherent case with the noetherian case.

The results in this paper are from the author's thesis done at Brandeis University under Professor Maurice Auslander, whom the author wishes to thank for his encouragement, suggestions, and inspiration.

1. **Coherent functors and filtered limits.** In this section, $\mathfrak{A}$ will denote an abelian category with enough projectives, and $[\mathfrak{A}, \boldsymbol{Ab}]$ the abelian category of additive functors from $\mathfrak{A}$ to the category of abelian groups. (We assume always that our

Received by the editors May 12, 1967.

categories are U-categories for some Grothendieck universe U; see [6].) For A an object of $\mathfrak{A}$, we let $h_A = \mathfrak{A}(A, \ \ ) = \mathrm{Hom}_{\mathfrak{A}}(A, \ \ )$ denote the corresponding representable functor in $[\mathfrak{A}, \boldsymbol{Ab}]$.

A functor F of $[\mathfrak{A}, \boldsymbol{Ab}]$ is *coherent* if there is a morphism $A_0 \to A_1$ in $\mathfrak{A}$ and an exact sequence $h_{A_1} \to h_{A_0} \to F \to 0$ in $[\mathfrak{A}, \boldsymbol{Ab}]$.

We summarize some of the properties of coherent functors, given in [2], in the following

PROPOSITION 1.1. (a) *If F_1 and F_2 are coherent, then $F_1 \oplus F_2$ is coherent.*

(b) *If $F_1 \to F_2 \to F_3 \to 0$ is exact and F_1 and F_2 are coherent, then F_3 is coherent.*

(c) *Let F be coherent, and let $0 \to A_3 \to A_0 \to A_1 \to A_2 \to 0$ be exact in $\mathfrak{A}$ such that there is an exact sequence $h_{A_1} \to h_{A_0} \to F \to 0$ in $[\mathfrak{A}, \boldsymbol{Ab}]$. Then the zeroth right derived functor of F, R^0F, is naturally equivalent to h_{A_3}.*

(d) *If F is coherent and $R^0F = 0$, then for any half exact functor G in $[\mathfrak{A}, \boldsymbol{Ab}]$,* $\mathrm{Ext}^1(F, G) = 0$.

Let I be a small category. We say that I is a *filtered* category if the following properties hold (see §1 of Chapter 1 of [1]):

(F1) For morphisms $i \to j$ and $i \to j'$ in I, there are morphisms $j \to k$ and $j' \to k$ in I such that the resulting square diagram commutes.

(F2) Given two objects i and j of I and two morphisms $a_i: i \to j$ and $a_2: i \to j$ in I, there is a morphism $b: j \to k$ in I so that $ba_1 = ba_2$.

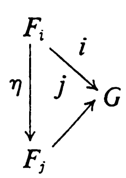
(F3) Given two objects i and i' of I, there is an object k of I and morphisms $i \to k$ and $i' \to k$ in I.

A *filtered system* in a category C is a functor $f: I \to C$, where I is a filtered category, and a *filtered limit* of a filtered system $f: I \to C$ is rt. $\lim_I f$, the right limit (or right root; see [5]) of f. If we let C_i denote $f(i)$, then we may denote this filtered system by $(C_i)_I$ and its limit by rt. $\lim_I C_i$.

Artin, in [1], has shown that filtered limits in $\boldsymbol{Ab}$ preserve exactness, and it is easy to extend this property to filtered limits in $[\mathfrak{A}, \boldsymbol{Ab}]$.

A full subcategory J of a category I is *cofinal* if, for every object i of I, there is an object j of J and a morphism $i \to j$ in I. It is clear that a cofinal subcategory of a filtered category is filtered. Artin ([1]) has shown that if $f: I \to \boldsymbol{Ab}$ is a filtered system and J is cofinal in I, then rt. $\lim_I f \cong$ rt. $\lim_J f/J$. This result extends to filtered systems in $[\mathfrak{A}, \boldsymbol{Ab}]$.

Let G be a functor in $[\mathfrak{A}, \boldsymbol{Ab}]$. Let $\mathfrak{A}_0/G$ be the category whose objects are natural transformations $i: F_i \to G$ in $[\mathfrak{A}, \boldsymbol{Ab}]$, where F_i is coherent, and whose morphisms are commutative triangles



in $[\mathfrak{A}, \boldsymbol{Ab}]$. Let $f: \mathfrak{A}_0/G \to [\mathfrak{A}, \boldsymbol{Ab}]$ be the obvious functor $f(i) = F_i$.

Lemma 1.2. (a) $\mathfrak{A}_0/G$ *is a filtered category, and*
(b) rt. lim $\mathfrak{A}_0/G\, f = G$.

The proof of part (a) is an easy application of parts (a) and (b) of Proposition 1.1, while part (b) follows from the fact that G may be written as a direct limit of a directed subsystem of f, and from this one can show that G satisfies the proper universal mapping property for the whole system.

The result we wish to obtain is the following:

Theorem 1.3. *Every half exact functor* $G: \mathfrak{A} \to \boldsymbol{Ab}$ *is the filtered limit of a filtered system of coherent half exact functors.*

Proof. The theorem is demonstrated if we can show that the full subcategory of $\mathfrak{A}_0/G$ whose objects are natural transformations $i: F_i \to G$, where F_i is half exact, is cofinal in $\mathfrak{A}_0/G$. This is a consequence of the following

Lemma 1.4. *For any coherent functor F, there is a coherent half exact functor F' and a monomorphism $\eta: F \to F'$ which gives an isomorphism $R^0\eta: R^0F \cong R^0F'$. Furthermore, for any half exact functor G and natural transformation $i: F \to G$, there is a natural transformation $i': F' \to G$ such that $i'\eta = i$.*

Proof. Let $A_0 \to A_1$ be a morphism in $\mathfrak{A}$ with $h_{A_1} \to h_{A_0} \to F \to 0$ exact in $[\mathfrak{A}, \boldsymbol{Ab}]$. Let $0 \to A_3 \to A_0 \to A_1 \to A_2 \to 0$ be exact in $\mathfrak{A}$. Let $P \to A_1$ be an epimorphism in $\mathfrak{A}$ with P projective, let

$$\begin{array}{ccc} C & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ A_0 & \longrightarrow & A_1 \end{array}$$

be a pull-back diagram in $\mathfrak{A}$, and let $0 \to K \to P \to A$ be exact in $\mathfrak{A}$. Then we have a commutative diagram

$$(1)\qquad \begin{array}{ccccccccccc} & & & & 0 & & 0 & & & & \\ & & & & \big\downarrow & & \big\downarrow & & & & \\ & & & & K & = & K & & & & \\ & & & & \big\downarrow & & \big\downarrow & & & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & C & \longrightarrow & P & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & \| & & \big\downarrow & & \big\downarrow & & \| & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & 0 \\ & & & & \big\downarrow & & \big\downarrow & & & & \\ & & & & 0 & & 0 & & & & \end{array}$$

in $\mathfrak{A}$ whose rows and columns are exact.

Therefore, letting F' be the cokernel of $h_P \to h_C$ in $[\mathfrak{A}, \boldsymbol{Ab}]$, diagram (1) induces a commutative diagram

$$
(2) \qquad
\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
h_{A_1} & \longrightarrow & h_{A_0} & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
h_P & \longrightarrow & h_C & \longrightarrow & F' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
h_K & = & h_K & & & &
\end{array}
$$

in $[\mathfrak{A}, \boldsymbol{Ab}]$ whose rows and columns are exact.

Diagram (2) induces a natural transformation $\eta\colon F \to F'$ which is easily seen to be a monomorphism. Since P is projective, h_P is exact, so F' is a coherent half exact functor. By Proposition (1.1c), $R^0F \cong h_{A_3} \cong R^0F'$.

Let $0 \to F \to F' \to T \to 0$ be exact in $[\mathfrak{A}, \boldsymbol{Ab}]$. By Proposition (1.1b), T is coherent, and since R^0 is exact and $R^0F \cong R^0F'$, $R^0T=0$.

Suppose G is a half exact functor in $[\mathfrak{A}, \boldsymbol{Ab}]$. Then we have an exact sequence

$$[F', G] \to [F, G] \to \operatorname{Ext}^1(T, G)$$

(where $[F, G]$ denotes the group of natural transformations from F to G). But by Proposition (1.1d), $\operatorname{Ext}^1(T, G)=0$, so that $\eta^*\colon [F', G] \to [F, G]$ is an epimorphism.

2. Tor. In this section, R will denote a ring which is both left and right noetherian.

Let $G\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ be a half exact functor. Let us restrict G to a functor G' on ${}_R\boldsymbol{M}'$, the full subcategory of ${}_R\boldsymbol{M}$ whose objects are the finitely generated modules. Since R is noetherian, ${}_R\boldsymbol{M}'$ is an abelian subcategory of ${}_R\boldsymbol{M}$, and it has enough projectives. Therefore, by Theorem 1.3, $G' = \text{rt.}\lim_I F_i'$, where $(F_i')_I$ is a filtered system of coherent half exact functors in $[{}_R\boldsymbol{M}', \boldsymbol{Ab}]$.

From the proof of Lemma 1.4, we see that we may assume that for each object i of I, we have an exact sequence

$$h'_{P_i} \to h'_{C_i} \to F_i' \to 0$$

in $[{}_R\boldsymbol{M}', \boldsymbol{Ab}]$ (h'_A denotes the restriction of $h_A = \operatorname{Hom}_R(A, \quad)$ to finitely generated modules), where C_i is a finitely generated module and P_i is a finitely generated projective R-module.

We extend F_i' to a functor F_i in $[{}_R\boldsymbol{M}, \boldsymbol{Ab}]$ by letting F_i be defined by the exact sequence of functors $h_{P_i} \to h_{C_i} \to F_i \to 0$. Since h_{P_i} is exact and since P_i and C_i are finitely presented modules, it follows that F_i is a coherent half exact functor which preserves direct limits.

The filtered system $(F_i')_I$ in $[_RM', Ab]$ clearly extends to a filtered system $(F_i)_I$ in $[_RM, Ab]$, and since every R-module may be written as a direct limit of finitely generated R-modules, we have the following

Proposition 2.1. *If G: $_RM \to Ab$ is a half exact functor which preserves direct limits, then $G = \text{rt. lim}_I F_i$, where $(F_i)_I$ is a filtered system of coherent half exact functors in $[_RM, Ab]$ which preserve direct limits.*

Let us assume now that G is half exact and preserves direct limits, and that we have G written as the limit of a filtered system $(F_i)_I$ as in the conclusion of Proposition 2.1. By Theorem 7.5 in [2], we have, for each object i of I, an exact sequence

$$(*) \qquad T_2(F_i) \to F_i(R) \otimes_R \to F_i \to T_1(F_i) \to 0,$$

where there is a right module N_i for which $T_1(F_i) = \text{Tor}_1^R(N_i, \quad)$ and $T_2(F_i) = \text{Tor}_2^R(N_i, \quad)$.

Let us make some remarks about these sequences.

Remark 1. In our case, for each object i of I, we have an exact sequence $h_{P_i} \to h_{C_i} \to F_i \to 0$, where C_i is finitely generated. In the proof of Theorem 7.5 of [2], N_i is constructed by choosing an exact sequence $P_1 \to P_0 \to C_i \to 0$, where P_0 and P_1 are projective R-modules, and requiring that $P_0^* \to P_1^* \to N_i \to 0$ be exact. Since C_i is finitely generated in our case, we may further require N_i to be finitely generated.

Remark 2. Given a natural transformation $F_i \to F_j$, we may construct a commutative diagram

$$\begin{array}{ccccccc} h_{P_i} & \longrightarrow & h_{C_i} & \longrightarrow & F_i & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ h_{P_j} & \longrightarrow & h_{C_j} & \longrightarrow & F_j & \longrightarrow & 0 \end{array}$$

since h_A is projective in $[_RM, Ab]$ for each R-module A. It is clear from the proof of Theorem 7.5 in [2] that from the R-homomorphism $C_j \to C_i$ which induces $h_{C_i} \to h_{C_j}$, we may construct an R-homomorphism f_{ij}: $N_i \to N_j$ so that the induced maps

$$\text{Tor}_p^R(f_{ij}, \quad): \text{Tor}_p^R(N_i, \quad) \to \text{Tor}_p^R(N_j, \quad), \qquad p = 1 \text{ or } 2,$$

give a commutative diagram

$$\begin{array}{ccccccccc} T_2(F_i) & \longrightarrow & F_i(R) \otimes_R & \longrightarrow & F_i & \longrightarrow & T_1(F_i) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ T_2(F_j) & \longrightarrow & F_j(R) \otimes_R & \longrightarrow & F_j & \longrightarrow & T_1(F_j) & \longrightarrow & 0. \end{array}$$

From Remark 2, it follows that the sequences (*) are functorial in F_i, so that we may take their filtered limit and obtain the exact sequence

$$(**) \qquad \text{rt. lim}_I T_2(F_i) \to G(R) \otimes_R \to G \to \text{rt. lim}_I T_1(F_i) \to 0.$$

Now, our characterization of Tor-functors depends on showing that a filtered limit of Tor-functors $(T_i)_I$ of the form $T_i = \mathrm{Tor}_1^R(N_i,\ \ )$, N_i a finitely generated right R-module, is a Tor-functor.

Suppose we had a filtered system $(M_j)_J$ of modules in $\boldsymbol{M}_R$ so that the system $(\mathrm{Tor}_1^R(M_j,\ \ ))_J$ was equivalent to the given system of Tor-functors $(T_i)_I$ (we may say $(M_j)_J$ *represents* the system $(T_i)_I$). Then it is easy to show that rt. $\lim_I T_i = \mathrm{Tor}_1^R(\text{rt.} \lim_J M_j,\ \ )$. So the problem now is representing a filtered system of Tor-functors by a filtered system of modules in $\boldsymbol{M}_R$.

Let $\Pi(\boldsymbol{M}_R') = \Pi_R$ be the homotopy category of $\boldsymbol{M}_R'$, i.e., that category whose objects are finitely generated right R-modules and, given two such modules M and N, $\Pi_R(M, N)$ is $\mathrm{Hom}_R(M, N)$ *modulo* the subgroup of R homomorphisms from M to N which factor through a projective R-module. It is not too difficult to establish the following

LEMMA 2.2. *If S and T are functors from which $S = \mathrm{Tor}_1^R(M,\ \ )$ and $T = \mathrm{Tor}_1^R(N,\ \ )$, M and N finitely generated right R-modules, then the map*

$$\mathrm{Hom}_R(M, N) \to [\mathrm{Tor}_1^R(M,\ \ ), \mathrm{Tor}_1^R(N,\ \ )]$$

defined by $f \to \mathrm{Tor}_1^R(f,\ \ )$ induces an isomorphism $\Pi_R(M, N) \cong [S, T]$, where $[S, T]$ denotes the group of natural transformations from S to T.

Therefore, our filtered system of Tor-functors $(T_i)_I$ is represented by a filtered system $(N_i)_I$ in Π_R.

Now the problem is "lifting" a filtered system in Π_R to a filtered system in $\boldsymbol{M}_R'$. We have not been able to do this except in the case where R is of global dimension 1.

For the rest of this section, we suppose that gl. dim. $\mathrm{R} = 1$.

Let M be a finitely generated right R-module. Let tM be the kernel of the canonical map $M \to M^{**}$ of a module M to its double dual, and let pM be the image of this map. Since $M \to M^{**}$ is functorial in M, t and p may be considered as functors.

Since the dual of M, M^*, is finitely generated, there is an epimorphism $P \to M^*$, where P is a finitely generated projective R-module. Therefore, we have a series of inclusions $pM \to M^{**} \to P^*$. Since P^* is projective and gl. dim. $R = 1$, pM is projective, so that the exact sequence

$$0 \to tM \to M \to pM \to 0$$

splits.

Using this fact, it is easy to demonstrate

LEMMA 2.3. *The map $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(tM, tN)$, defined by $f \mapsto tf$, induces an isomorphism $\Pi_R(M, N) \cong \mathrm{Hom}_R(tM, tN)$.*

Since the map $tM \to M$ becomes an isomorphism in Π_R, it follows from Lemma 2.3 that any filtered system $(N_i)_I$ in Π_R lifts to a filtered system $(tN_i)_I$ in $\boldsymbol{M}_R'$.

We state our conclusions in a theorem (we note that when gl. dim. $R=1$, $\mathrm{Tor}_2^R(N,\ \ )=0$).

THEOREM 2.4. *Suppose R is a noetherian ring of global dimension 1, and $G\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ is a half exact functor which preserves direct limits. Then there is a right R-module N and an exact sequence*

$$0 \to G(R) \otimes_R \to G \to \mathrm{Tor}_1^R(N,\ \ ) \to 0.$$

COROLLARY 2.5. *Suppose R is a ring satisfying the hypothesis of Theorem 2.4. Then a functor $G\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ is a* Tor*-functor if and only if G is half exact, G preserves direct limits, and $G(R)=0$.*

3. **Coherent rings and** Tor. Let R and ${}_R\boldsymbol{M}$ be as before, and let ${}_R\boldsymbol{M}'$ denote the full subcategory of ${}_R\boldsymbol{M}$ whose objects are the finitely presented left modules of R. Let $i\colon {}_R\boldsymbol{M}' \to {}_R\boldsymbol{M}$ denote the inclusion functor. Then i induces a functor

$$i^*\colon [{}_R\boldsymbol{M}, \boldsymbol{Ab}] \to [{}_R\boldsymbol{M}', \boldsymbol{Ab}],$$

which has a left adjoint

$$u\colon [{}_R\boldsymbol{M}', \boldsymbol{Ab}] \to [{}_R\boldsymbol{M}, \boldsymbol{Ab}].$$

Let $v\colon [{}_R\boldsymbol{M}, \boldsymbol{Ab}] \to [{}_R\boldsymbol{M}, \boldsymbol{Ab}]$ be the composition $v=u \circ i^*$.

If I denotes the identity functor on $[{}_R\boldsymbol{M}, \boldsymbol{Ab}]$ and I' the identity functor on $[{}_R\boldsymbol{M}', \boldsymbol{Ab}]$, then from the fact that u is a left adjoint of i it follows that we have natural transformations $\phi\colon I' \to i^* \circ u$ and $\theta\colon v=u \circ i^* \to I$. In our case, ϕ is a natural equivalence of functors, and the following can be proved.

THEOREM 3.1. (a) *v is exact and preserves right limits,*

(b) *for any functor $F\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$, $vF\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ preserves direct limits, and for any natural transformation $\eta\colon L \to F$, where $L\colon {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ preserves direct limits, there is a unique natural transformation $\eta'\colon L \to vF$ so that $\theta_F \circ \eta' = \eta$.*

We do not give a proof of this theorem here, as it can be proved in a more general context. However, we may define more explicitly the functor v as follows: Let F be any functor in $[{}_R\boldsymbol{M}, \boldsymbol{Ab}]$ and let M be any left R-module. Then there is a directed system $(M_i')_I$ of finitely presented left R-modules for which rt. $\lim_I M_i' = M$. We let $vF(M) = $ rt. $\lim_I F(M_i')$. Once it is shown that v is a well-defined functor, the proof of the above theorem is trivial.

Let C be a left R-module and let C^* denote the right R-module $C^* = \mathrm{Hom}_R(C, R)$, the dual of C. Then $C^* \otimes_R$ may be considered as a functor in $[{}_R\boldsymbol{M}, \boldsymbol{Ab}]$. Let us consider the following facts.

(1) For any left R-module C, there is a canonical natural transformation $\lambda\colon C^* \otimes_R \to \mathrm{Hom}_R(C,\ \ ) = h_C$ defined by $\lambda_M(f \otimes m)(c) = f(c) \cdot m$ for any left

R-module M and elements m in M, c in C, and f in C^*. Since $C^* \otimes_R$ preserves direct limits, we have a unique natural transformation

$$\lambda': C^* \otimes_R \to vh_C$$

for which $\theta_{h_C} \circ \lambda' = \lambda$.

(2) $\lambda'_R: C^* \otimes_R R \to vh_C(R)$ is easily seen to be essentially the identity map on C^*. Since $C^* \otimes_R$ and vh_C preserve direct limits (and therefore direct sums), it follows that λ'_P is an isomorphism for any left projective R-module P. Since $C^* \otimes_R$ is right exact, we have shown that $C^* \otimes_R$ is the zeroth left derived functor of vh_C, or in symbols, $L_0(vh_C) \cong C^* \otimes_R$.

(3) Let P be a projective left R-module. Every short exact sequence of left R-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ may be written as a direct limit of a directed system $(M'_{1i} \to M'_{2i} \to M'_{3i} \to 0)_I$, where for each $i \in I$, $M'_{1i} \to M'_{2i} \to M'_{3i} \to 0$ is an exact sequence of finitely presented left R-modules. Since h_P is right exact, and for every finitely presented left module M', $vh_P(M') \cong h_P(M')$, it follows from the above fact and the explicit definition of v that vh_P is right exact. Therefore, $P^* \otimes_R \cong L_0(vh_P) \cong vh_P$ for every projective R module P.

We summarize these facts in the following proposition:

PROPOSITION 3.2. *For every left R-module C, there is a natural transformation $\lambda': C^* \otimes_R \to vh_C$ induced by $\lambda: C^* \otimes_R \to h_C$, and λ' is an isomorphism when C is projective.*

Let us now proceed (as in [2]) to attempt to identify the kernel and cokernel of $\lambda': C^* \otimes_R \to vh_C$ as Tor functors for any left R-module C.

Let $P_1 \to P_0 \to C \to 0$ be an exact sequence in ${}_R M$, with P_0 and P_1 projective R-modules. Let N be the cokernel of $P_0^* \to P_1^*$, so that we have an exact sequence of right R-modules

$$0 \to C^* \to P_0^* \to P_1^* \to N \to 0.$$

Since $0 \to h_C \to h_{P_0} \to h_{P_1}$ is exact and v is exact by 3.1, we have an exact sequence

$$0 \to vh_C \to vh_{P_0} \to vh_{P_1}.$$

Proposition 3.2 gives us a commutative diagram

$$\begin{array}{ccccccccc}
 & & C^* \otimes_R & \longrightarrow & P_0^* \otimes_R & \longrightarrow & P_1^* \otimes_R & \longrightarrow & N \otimes_R \longrightarrow 0 \\
 & & \downarrow & & \wr\| & & \wr\| & & \\
0 & \longrightarrow & vh_C & \longrightarrow & vh_{P_0} & \longrightarrow & vh_{P_1} & &
\end{array}$$

Following [2], we wish to have the sequence $P_0^* \to P_1^* \to N \to 0$ a flat presentation of N, i.e., we wish R to have the property that for every projective left R-module P, P^* is a flat right R-module. This is equivalent to the assumption that R

is a *left coherent ring*. (We may adopt this as a definition, although the reader may see [3, Exercise 12, p. 63] for a definition and some properties of coherent rings.)

PROPOSITION 3.3. *If R is a left coherent ring, then for every left R-module C, there is a right R-module N and an exact sequence* $0 \to \mathrm{Tor}_2^R(N, \quad) \to C \otimes_R \to vh_C \to \mathrm{Tor}_1^R(N, \quad) \to 0$.

Once we have that $P_0^* \to P_1^* \to N \to 0$ is a flat presentation of N, the exact sequence follows immediately from the above diagram.

THEOREM 3.4. *Let R be a coherent ring, and let $h_P \to h_C \to F \to 0$ be exact, where C and P are left R-modules with P projective (so that F is a coherent half exact functor). Then there is a right R-module N and an exact sequence*

$$\mathrm{Tor}_2^R(N, \quad) \to F(R) \otimes_R \to vF \to \mathrm{Tor}_1^R(N, \quad) \to 0.$$

REMARK. This is an extension of Auslander's exact sequence, for if F also preserves direct limits, then $vF \cong F$. Our proof below is essentially an exact copy of that of Theorem 7.5 of [2].

Proof. Since $h_P \to h_C \to F \to 0$ is exact, we have an exact sequence of right R-modules $P^* \to C^* \to F(R) \to 0$, and therefore an exact sequence of functors $P^* \otimes_R \to C^* \otimes_R \to F(R) \otimes_R \to 0$.

Since v is exact, we also have an exact sequence of functors $vh_P \to vh_C \to vF \to 0$.

By the above, and by 3.3, we have a right module N which gives us a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \mathrm{Tor}_2^R(N, \quad) & & & & \\
 & & \downarrow & & & & \\
P^* \otimes_R & \longrightarrow & C^* \otimes_R & \longrightarrow & F(R) \otimes_R & \longrightarrow & 0 \\
\wr\| & & \downarrow & & & & \\
vh_P & \longrightarrow & vh_C & \longrightarrow & vF & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
 & & \mathrm{Tor}_1^R(N, \quad) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

with exact rows and columns. This diagram induces the desired exact sequence.

We remark that every noetherian ring is coherent, and that Prüfer rings and rank one valuation rings are examples of coherent rings which are not necessarily noetherian.

4. **Remarks on the problem of characterizing** Tor. Let R be a coherent ring, and let $G: {}_R\boldsymbol{M} \to \boldsymbol{Ab}$ be a half exact functor which preserves direct limits. Let us attempt to develop an exact sequence for G analogous to our procedure in §2.

By Theorem 1.3, $G \cong$ rt. $\lim_I F_i$ where $(F_i)_I$ is a filtered system of coherent half exact functors. By Theorem 3.1, v preserves right limits, so that $vG \cong$ rt. $\lim vF_i$. Since G preserves direct limits, $vG \cong G$, so that $G \cong$ rt. $\lim_I vF_i$.

For each i in I, we have an exact sequence

$$T_2(F_i) \to F_i(R) \otimes_R \to vF_i \to T_1(F_i) \to 0$$

(where there is a right R-module N_i with $T_p(F_i) = \mathrm{Tor}_p^R(N_i,\ \ )$ for $p=1, 2$) and, as in §2, these sequences are easily shown to be functorial in F_i.

Therefore, by taking limits, we obtain an exact sequence.

$$\text{rt. } \lim_I T_2(F_i) \to G(R) \otimes_R \to G \to \text{rt. } \lim_I T_1(F_i) \to 0,$$

which leaves us, as in the noetherian case, with the problem of showing that a limit of a filtered system $(T_i)_I$ of Tor functors is again a Tor functor.

However, the noetherian case seems easier, as we may assume that $T_i = \mathrm{Tor}(N_i,\ \ )$ for a finitely generated module N_i for each $i \in I$, where in the coherent case N_i need not be finitely generated. This is why we adopted a somewhat different method in §2 from that of §§3 and 4 (using the fact that ${}_R\boldsymbol{M}'$ is abelian when R is noetherian), since we were able to obtain the desired characterization of Tor at least for rings of global dimension 1.

References

1. M. Artin, *Grothendieck topologies* (Mimeographed notes), Harvard Univ., Cambridge, Mass., 1962.

2. M. Auslander, *Coherent functors*, Proc. Conf. Categorical Algebra, La Jolla, Calif., 1965, Springer-Verlag, New York, 1966.

3. N. Bourbaki, *Algèbre commutative*, Chapitres 1 et 2, Hermann, Paris, 1961.

4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

5. P. Freyd, *Abelian categories*, Harper and Row, New York, 1964.

6. P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

7. D. Newell, *Stability of categories and a characterization of* Tor, Thesis, Brandeis Univ., Waltham, Mass., 1966.

Tufts University,
Medford, Massachusetts